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The Numerical Solution of Plane Potential Problems by Improved Boundary Integral Equation Methods

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A modified boundary integral (BIE) method which facilitates accurate solution of Laplacian boundary-value problems is presented. This method is designed specifically for treatment of problems in which singularities occur on the interface between two regions with different physical properties, and is illustrated by application to two physical problems. Analytic expressions for the integrals arising in the piecewise-linear and piecewise-quadratic BIE approximations are also presented. These analytical expressions afford an appreciable reduction in computational time when compared with previously employed quadrature formulae.

INTRODUCTION

Elliptic boundary-value problems arising from the examination of physical situations encountered in engineering and mathematical physics are, in general, intractable by analytical treatment. Although various numerical techniques have been proposed for the solution of such problems, e.g., the finite difference [1], finite element [2], and boundary integral equation (BIE) [3] methods, standard forms of these techniques tend to yield inaccurate solutions for problems involving boundary singularities. Consequently, the possibility of modifying these numerical techniques to give special treatment to singular points, and thereby to obtain solutions which converge more rapidly has received considerable attention [3–10].

Symm [3] devised a modification of the BIE method which can successfully treat boundary singularities in two-dimensional Laplacian problems. The results obtained by employing this method offer considerable improvement over those given by Galerkin methods modified by either mesh refinement near the singularity, or inclusion of terms having the analytical form of the singularity [3, 6]. The present investigation considers problems in which the boundary singularities occur on the interface between two regions with different physical properties, e.g., different thermal conductivities in heat diffusion problems [11], and different dielectric permittivities in electromagnetics [4]. Blue [14] discussed how the BIE techniques may be applied to multiple-region problems and indicated that this would require a significant change in data structure in comparison with single-region problems. In this study these BIE

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techniques are implemented and then extended, in a manner analogous to the modification devised by Symm [3], to incorporate the analytical form of the singularity and thereby to facilitate a more accurate solution. In order to illustrate the solution capailities of this singular BIE method two problem which involve L-shaped domains with mixed boundary conditions are examined. Solutions are cntrasted with those obtained by employing standard piecewise-constant, piecewise-linear and piecewise-quadratic BIE implementations. Furthermore, previously undetermined analytical solutions for the integrals associated with the piecewise-linear and piecewise-quadratic BIE formulations are presented. The use of these analytical expressions instead of quadrature formulae [13] not only reduces the programming complexity but also results in substantial reductions in the computational time.

The Standard BIE Methods

As detailed descriptions of the various BIE formulations for obtaining solutions to plane potential boundary-value problems have previously been presented [3, 12, 13], only those features necessary to facilitate a concise explanation of the proposed modifications, are presented in this study.

For any sufficiently smooth function ϕ which satisfies Laplace's equation in a plane domain Ω, having a piecewise-smooth boundary $\partial\Omega$, Green's Integral Formula may be expressed as

$$\int_{\partial\Omega} \{\phi(q) \log' |p-q| - \phi'(q) \log |p-q|\}\, dq = \eta(p)\, \phi(p), \tag{1}$$

where

(i) $p \in \Omega + \partial\Omega$, $q \in \partial\Omega$.

(ii) dq denotes the differential increment of $\partial\Omega$ at q.

(iii) The prime $'$ denotes the derivative in the direction of the outward normal to $\partial\Omega$ at q.

(iv) If $p \in \Omega$ then $\eta = 2\pi$, and if $p \in \partial\Omega$ then η is the internal angle included between the tangents to $\partial\Omega$ on either side of p.

If either ϕ, ϕ' or a linear combination of ϕ and ϕ' is prescribed at each point of $\partial\Omega$, then solution of the equation

$$\int_{\partial\Omega} \{\phi(q) \log' |\bar{q}-q| - \phi'(q) \log |\bar{q}-q|\}\, dq - \eta(\bar{q})\, \phi(\bar{q}) = 0, \qquad q, \bar{q} \in \partial\Omega, \tag{2}$$

determines ϕ and ϕ' at each point of $\partial\Omega$. The potential ϕ at any point $p \in (\Omega + \partial\Omega)$ can then be computed employing Green's Integral Formula, Eq. (1).

Thus, application of Green's Boundary Formula, Eq. (2), enables well-posed two-dimensional Laplacian boundary-value problems to be reformulated as integral equations in which the unknowns are boundary values of the potential, ϕ, and its normal derivative, ϕ', complementary to those prescribed by the boundary conditions. However, in practice these integral equations can rarely be solved analytically, and therefore various numerical techniques have been proposed to enable application of Green's Boundary Formula [3, 12, 13].

In the classical BIE (CBIE) method [3], the boundary $\partial\Omega$ is subdivided into smooth intervals, $\partial\Omega_j, j = 1,..., N$, on which ϕ and ϕ' are approximated by piecewise-constant functions ϕ_j and ϕ'_j. Application of the corresponding discretized form of the Integral Formula,

$$\sum_{j=1}^{N} \left\{ \phi_j \int_{\partial\Omega_j} \log' |p-q| \, dq - \phi'_j \int_{\partial\Omega_j} \log |p-q| \, dq \right\} = \eta(p)\,\phi(p),$$
$$p \in \Omega + \partial\Omega, \quad q \in \partial\Omega, \quad (3)$$

to the midpoint, $p \equiv q_j$, of each interval and enforcing the boundary conditions, generates a system of linear algebraic equations. Solution of these equations determines ϕ_j and ϕ'_j on each interval. The solution at any interior point can then be computed by a relatively simple quadrature, Eq. (3).

The linear BIE (LBIE) method, affords a slightly more sophisticated approximation of Green's Integral Formula than the classical BIE method. On each interval $\partial\Omega_j, j = 1,..., N$, ϕ and ϕ' are approximated by piecewise-linear functions

$$\phi = (1-\xi)\,\phi(q_j) + \xi\phi(q_{j+1}),$$
$$\phi' = (1-\xi)\,\phi'(q_j) + \xi\phi'(q_{j+1}),$$

where q_j and q_{j+1} are the endpoints of $\partial\Omega_j$, and ξ is a linear function which increases from zero at q_j to unity at q_{j+1}. Correspondingly, Green's Integral Formula becomes

$$\sum_{j=1}^{N} \left\{ \phi_j \int_{\partial\Omega_j} (1-\xi) \log' |p-q| \, dq + \phi_{j+1} \int_{\partial\Omega_j} \xi \log' |p-q| \, dq \right\}$$
$$- \sum_{j=1}^{N} \left\{ \phi'_j \int_{\partial\Omega_j} (1-\xi) \log |p-q| \, dq + \phi'_{j+1} \int_{\partial\Omega_j} \xi \log |p-q| \, dq \right\}$$
$$= \eta(p)\,\phi(p), \qquad p \in \Omega + \partial\Omega, \quad q \in \partial\Omega, \quad (4)$$

where ϕ_j and ϕ'_j denote $\phi(q_j)$ and $\phi'(q_j)$, respectively. A system of linear algebraic equations in the unknown ϕ_j and ϕ'_j can now be generated by collocating Eq. (4) at each of the points $p \equiv q_j$.

A more accurate approximation of the solution to the boundary integral equations can be obtained using the quadratic BIE (QBIE) method [13]. In this approach, on each interval $\partial\Omega_j$, $j=1,\ldots,N$, ϕ and ϕ' are approximated by piecewise-quadratic functions,

$$\phi = M_1(\xi)\,\phi(q_{2j-1}) + M_1(\xi)\,\phi(q_{2j}) + M_3(\xi)\,\phi(q_{2j+1}),$$
$$\phi' = M_1(\xi)\,\phi'(q_{2j-1}) + M_2(\xi)\,\phi'(q_{2j}) + M_3(\xi)\,\phi'(q_{2j-1}),$$

where q_{2j-1} and q_{2j+1} are the endpoints of $\partial\Omega_j$, q_{2j} is the midpoint of $\partial\Omega_j$, ξ is a linear function which increases from zero at q_{2j-1} to unity at q_{2j+1}, and

$$M_1(\xi) = 1 - 3\xi + 2\xi^2,$$
$$M_2(\xi) = 4\xi - 4\xi^2,$$
$$M_3(\xi) = -\xi - 2\xi^2.$$

On the basis of these approximations the Integral Formula becomes

$$\begin{aligned}
&\sum_{j=1}^{N}\left\{\phi_{2j-1}\int_{\partial\Omega_j} M_1(\xi)\log'|p-q|\,dq + \phi_{2j}\int_{\partial\Omega_j} M_2(\xi)\log'|p-q|\,dq\right.\\
&\qquad\left.+\,\phi_{2j+1}\int_{\partial\Omega_j} M_3(\xi)\log'|p-q|\,dq\right\}\\
&\qquad-\sum_{j=1}^{N}\left\{\phi'_{2j-1}\int_{\partial\Omega_j} M_1(\xi)\log|p-q|\,dq + \phi'_{2j}\int_{\partial\Omega_j} M_2(\xi)\log|p-q|\,dq\right.\\
&\qquad\left.+\,\phi'_{2j+1}\int_{\partial\Omega_j} M_3(\xi)\log|p-q|\,dq\right\}\\
&\qquad=\eta(p)\,\phi(p), \qquad p\in\Omega+\partial\Omega, \quad q\in\partial\Omega.
\end{aligned} \tag{5}$$

A system of linear algebraic equations in the unknown ϕ_j and ϕ'_j can now be generated by applying formula (5) to each of the points $p \equiv q_j$, $j=1,\ldots,2N$. Thus, for an N interval discretization, the QBIE method requires solution of $2N$ equations in $2N$ unknowns, whereas the CBIE and LBIE methods require solution of N equations in N unknowns.

With the classical BIE formulation nodal points are situated only at segment midpoints and therefore ϕ' has precisely one value at each of these nodal points. However, with the linear and quadratic BIE formulations, nodes are situated at segment endpoints and therefore at domain corners ϕ' has two components; one related to each of the sides adjacent to the corner. Thus, the linear and quadratic BIE methods are restricted to problems for which a relation of the form

$$\phi' = \alpha\phi + \beta \qquad (\alpha, \beta \text{ given functions})$$

is prescribed on at least one of the sides of the corner.

If the interval $\partial\Omega_j$ is a straight-line segment, then the integrals in formulae (3), (4) and (5) can be evaluated exactly using

$$\int_{\partial\Omega_j} \log' |p-q|\, dq = I_1, \tag{6}$$

$$\int_{\partial\Omega_j} \log |p-q|\, dq = I_2, \tag{7}$$

$$\int_{\partial\Omega_j} \xi \log' |p-q|\, dq = \frac{1}{h}(a\cos\beta I_1 + I_2), \tag{8}$$

$$\int_{\partial\Omega_j} \xi \log |p-q|\, dq = \frac{1}{h}(a\cos\beta J_1 + J_2), \tag{9}$$

$$\int_{\partial\Omega_j} \xi^2 \log' |p-q|\, dq = \frac{1}{h^2}((h-2a\cos\beta)^2 I_1 - 4(h-2a\cos\beta) I_2 + 4I_3), \tag{10}$$

$$\int_{\partial\Omega_j} \xi^2 \log |p-q|\, dq = \frac{1}{h^2}((h-2a\cos\beta)^2 J_1 - 4(h-2a\cos\beta) J_2 + 4J_3), \tag{11}$$

where

$$I_1 = \psi, \tag{12}$$

$$I_2 = a\sin\beta(\log b - \log a), \tag{13}$$

$$I_3 = a\sin\beta(h - a\psi\sin\beta), \tag{14}$$

$$J_1 = a\cos\beta(\log a - \log b) + h(\log b - 1) + a\psi\sin\beta; \tag{15}$$

$$J_2 = \tfrac{1}{2}(b^2\log b - a^2\log a) - \tfrac{1}{4}(b^2 - a^2), \tag{16}$$

$$J_3 = \tfrac{1}{3}\{(h - a\cos\beta)^3(\log b - \tfrac{1}{3}) + (a\cos\beta)^3(\log a - \tfrac{1}{3}) + (a\sin\beta)^2(h - a\psi\sin\beta)\} \tag{17}$$

and if q_{aj} and q_{bj} denote the endpoints of $\partial\Omega_j$, Fig. 1, then a, b and h are the lengths of the lines joining p to q_{aj}, p to q_{bj} and q_{aj} to q_{bj}, respectively, and β and ψ are the angles $q_{bj}q_{aj}p$ and $q_{aj}pq_{bj}$, respectively.

The analytical solutions for the integrals associated with the CBIE method, Eqs. (6) and (7), were presented by Symm [3]. However, the integrals associated with the LBIE and QBIE methods, Eqs. (4) and (5), have previously been evaluated numerically [13]. Evaluation of these integrals by the analytical expressions, (6)–(11), requires only a fraction of the computational time taken by an accurate numerical technique, and since for an N interval discretization each of the integrals has to be evaluated N times for every point to which Green's Integral Formula is applied, it is apparent that these analytical expressions yield appreciable reductions in the computational times required by the LBIE and QBIE methods.

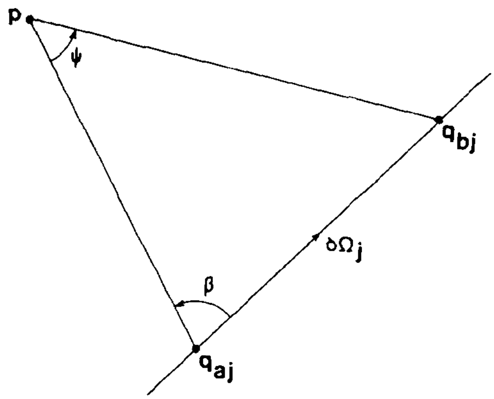

FIG. 1. Straight-line segment geometry.

To demonstrate the problems caused by the presence of boundary singularities, the CBIE, LBIE and QBIE methods are applied to two physical problems involving L-shaped domains.

Problem 1

This problem arises from the examination of heat flow through finned surfaces [11], and involves an L-shaped composite of two rectangular domains, Fig. 2, having different thermal conductivities. The temperature distribution ϕ within the domain $(A+B)$, Fig. 2, is determined by simultaneously solving

$$\nabla^2\phi_A = 0 \qquad \text{in region } A \tag{18}$$

and

$$\nabla^2\phi_B = 0 \qquad \text{in region } B \tag{19}$$

subject to the boundary conditions

$$\text{on } OA \qquad -k_A\phi'_A = h_2\phi_A, \tag{20i}$$

$$\text{on } AB \qquad -k_A\phi'_A = h_2\phi_A, \tag{20ii}$$

$$\text{on } BC \qquad \phi'_A = 0, \tag{20iii}$$

$$\text{on } CO \qquad \phi_A = \phi_B, \tag{20iv}$$

$$\text{on } CO \qquad k_A\phi'_A = -k_B\phi'_B, \tag{20v}$$

$$\text{on } CD \qquad \phi'_B = 0, \tag{20vi}$$

$$\text{on } DE \qquad k_B\phi'_B = h_1(1-\phi_B), \tag{20vii}$$

$$\text{on } EF \qquad \phi'_B = 0, \tag{20viii}$$

$$\text{on } FO \qquad -k_B\phi'_B = h_2\phi_B, \tag{20ix}$$

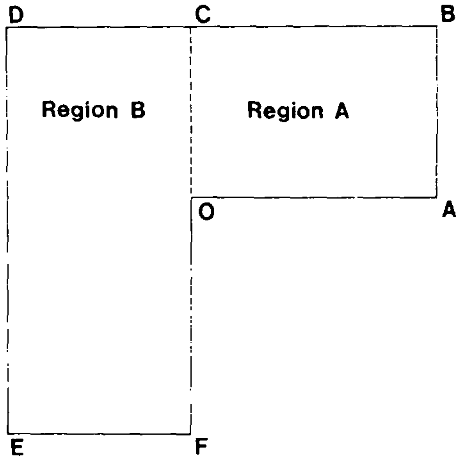

FIG. 2. L-Shaped domain

where k_A and k_B are the thermal conductivities of regions A and B, respectively, and h_1 and h_2 are the heat transfer coefficients at the surface DE and $FOAB$, respectively.

Applying Green's Boundary Formula, Eq. (2), to this problem gives rise to an integral equation involving two coupled contour integrals, one around $\partial\Omega_A$ $(= OABCO)$, and the other around $\partial\Omega_B$ $(= OCDEFO)$; the coupling arises through the interface boundary conditions (20iv) and (20v). Solution of this integral equation, by the numerical techniques described above, determines the boundary distributions of ϕ and ϕ'. Then to compute the potential ϕ at any interior point, it is only necessary to apply Green's Integral Formula to the boundary of the region in which that point lies. In particular, the potential at points on the common interface OC can be evaluated by applying Green's Integral Formula to either $\partial\Omega_A$ or $\partial\Omega_B$.

One of the quantities of physical importance in this problem is the rate of heat transfer, Q, which is given by [11],

$$Q = h_1 \int_{DE} (1 - \phi_B(q))\, dq \tag{21}$$

$$\equiv h_2 \int_{FO} \phi_B(q)\, dq + \int_{OAB} \phi_A(q)\, dq \tag{22}$$

as there are no heat sources situated within the domain $(A + B)$. It is apparent from expressions (21) and (22) that evaluation of Q only requires the boundary distribution of ϕ, and this is precisely the information obtained when the boundary integral equation representing the problem described by Eqs. (18), (19) and (20) is solved.

Results have been obtained by application of the CBIE, LBIE and QBIE methods, employing 50, 100 and 200 equal length boundary intervals, for the case $OA = AB =$

TABLE I

	Intervals		
	50	100	200
	i. Problem 1: CBIE method results		
Q_{IN}/Q_{OUT}	5.7387	5.7346	5.7330
Q_{OUT}	5.7342	5.7321	5.7317
Q_{IN}/Q_{OUT}	1.0007	1.0004	1.0002
	ii. Proble 1: LBIE method results		
Q_{IN}	5.7084	5.7199	5.7260
Q_{OUT}	5.7387	5.7350	5.7334
Q_{IN}/Q_{OUT}	0.9947	0.9973	0.9987
	iii. Problem 1: QBIE method results		
Q_{IN}	5.7129	5.7231	5.7280
Q_{OUT}	5.7334	5.7325	5.7321
Q_{IN}/Q_{OUT}	0.9964	0.9984	0.9993
	iv. Problem 1: MBIE method results		
Q_{IN}	5.7325	5.7321	5.7321
Q_{OUT}	5.7330	5.7325	5.7321
Q_{IN}/Q_{OUT}	0.9999	1.0000	1.0000

$EF = FO = 1$, $h_1 = 1000$, $h_2 = 10$, $k_A = 250$ and $k_B = 10$. This represents a heat exchanger comprised of copper in region A and steel in region B, with forced convection of water along DE and free convection of air around $FOAB$. In Table Ii–iii, Q_{IN} and Q_{OUT} represent the heat transfer rates corresponding to expressions (21) and (22), respectively; as the CBIE, LBIE and QBIE methods are based on assumed boundary variations of ϕ and ϕ', they need not give the same values for Q_{IN} and Q_{OUT}, although obviously a satisfactory solution must do so.

This problem has a singularity at the re-entrant corner O, and the results displayed in Table Ii–iii clearly illustrate the slow convergence caused by the presence of this singularity.

Problem 2

This problem arises in the study of plane potential flow through a porous medium between impervious pins [3, 5], and involves an L-shaped domain, Fig. 2, with a singularity at the re-entrant corner O. The determination of the potential ϕ requires solution of

$$\nabla^2\phi = 0 \qquad \text{in region } (A + B) \tag{23}$$

subject to the boundary conditions

$$\text{on } OA \quad \phi' = 0, \tag{24i}$$

$$\text{on } AB \quad \phi = 0, \tag{24ii}$$

$$\text{on } BD \quad \phi' = 0, \tag{24iii}$$

$$\text{on } DE \quad \phi = 1, \tag{24iv}$$

$$\text{on } EF \quad \phi' = 0, \tag{24v}$$

$$\text{on } FO \quad \phi' = 0. \tag{24vi}$$

This problem is essentially a special case of the problem

$$\nabla^2 \phi_A = 0 \quad \text{in region } A, \tag{25}$$

$$\nabla^2 \phi_B = 0 \quad \text{in region } B \tag{26}$$

subject to the conditions

$$\text{on } OA \quad \phi'_A = 0, \tag{27i}$$

$$\text{on } AB \quad \phi_A = 0, \tag{27ii}$$

$$\text{on } BC \quad \phi'_A = 0, \tag{27iii}$$

$$\text{on } CO \quad \phi_A = \phi_B, \tag{27iv}$$

$$\text{on } CO \quad k_A \phi'_A = -k_B \phi'_B, \tag{27v}$$

$$\text{on } CD \quad \phi'_B = 0, \tag{27vi}$$

$$\text{on } DE \quad \phi_B = 1, \tag{27vii}$$

$$\text{on } EF \quad \phi_B = 0, \tag{27viii}$$

$$\text{on } FO \quad \phi'_B = 0. \tag{27ix}$$

Solutions to the problem described by Eqs. (25), (26) and (27) have been obtained by application of the CBIE, LBIE and QBIE methods, employing 50, 100 and 200 equal length boundary intervals, for the case $OA = AB = EF = FO = 5$ and $k_A = k_B = 1$. The results presented in Tables IIi, IIii and IIiii show the potential at the lattice points of a unit mesh. Slow convergence, particularly in the neighbourhood of the singularity, is again evident.

In the next section a modified BIE method is described which gives special treatment to singular points and thereby yields, in general, considerably more accurate solutions than those given by the CBIE, LBIE and QBIE methods.

TABLE IIi

Problem 2: CBIE Method Results

1.0064	0.9176	0.8310	0.7409	0.6458	0.5404	0.5562	0.4458	0.3364	0.2246	0.1110	−0.0087
1.0032	0.9170	0.8312	0.7419	0.6477	0.5453	0.5532	0.4456	0.3369	0.2256	0.1125	−0.0044
1.0016	0.9167	0.8313	0.7423	0.6484	0.5475	0.5515	0.4455	0.3371	0.2259	0.1131	−0.0022
D					*C*	*C*					*B*
1.0009	0.9176	0.8326	0.7436	0.6497	0.5495	0.5520	0.4466	0.3374	0.2256	0.1126	−0.0013
0.9999	0.9176	0.8329	0.7445	0.6509	0.5518	0.5517	0.4472	0.3382	0.2264	0.1133	0.0001
1.0000	0.9175	0.8331	0.7448	0.6513	0.5520	0.5520	0.4474	0.3384	0.2267	0.1136	0.0000
0.9998	0.9202	0.8382	0.7517	0.6588	0.5587	0.5587	0.4521	0.3410	0.2278	0.1139	0.0001
1.0000	0.9204	0.8386	0.7524	0.6599	0.5598	0.5599	0.4531	0.3420	0.2287	0.1145	0.0000
1.0000	0.9204	0.8387	0.7526	0.6602	0.5602	0.5602	0.4535	0.3423	0.2290	0.1147	0.0000
0.9999	0.9252	0.8482	0.7662	0.6756	0.5734	0.5733	0.4618	0.3465	0.2306	0.1151	0.0001
1.0000	0.9254	0.8485	0.7668	0.6767	0.5749	0.5749	0.4634	0.3479	0.2317	0.1157	0.0000
1.0000	0.9254	0.8486	0.7670	0.6770	0.5754	0.5754	0.4639	0.3483	0.2321	0.1160	0.0000
0.9999	0.9324	0.8630	0.7892	0.7050	0.5931	0.6026	0.4743	0.3522	0.2330	0.1157	−0.0013
1.0000	0.9325	0.8632	0.7896	0.7060	0.6006	0.6006	0.4768	0.3540	0.2345	0.1167	0.0001
1.0000	0.9325	0.8633	0.7897	0.7064	0.6015	0.6015	0.4776	0.3546	0.2349	0.1170	0.0000
1.0000	0.9411	0.8818	0.8210	0.7571	0.6845	0.6305	0.4820	0.3550	0.2338	0.1148	−0.0089
1.0000	0.9411	0.8818	0.8210	0.7566	0.6785	0.6430	0.4853	0.3570	0.2356	0.1167	−0.0045
1.0000	0.9412	0.8818	0.8210	0.7565	0.6744	0.6514	0.4864	0.3576	0.2361	0.1173	−0.0023
					O	*O*					*A*

1.0000	0.9504	0.9017	0.8557	0.8164	0.7982	$N = 50$
1.0000	0.9503	0.9016	0.8554	0.8158	0.7969	100
1.0000	0.9503	0.9015	0.8553	0.8155	0.7964	200
1.0001	0.9587	0.9194	0.8848	0.8594	0.8499	
1.0000	0.9586	0.9192	0.8845	0.8589	0.8492	
1.0000	0.9585	0.9191	0.8844	0.8588	0.8489	
1.0000	0.9651	0.9326	0.9054	0.8867	0.8802	
1.0000	0.9649	0.9323	0.9050	0.8863	0.8796	
1.0000	0.9649	0.9322	0.9048	0.8861	0.8794	
1.0004	0.9692	0.9406	0.9173	0.9019	0.8966	
1.0000	0.9689	0.9402	0.9169	0.9014	0.8960	
1.0000	0.9688	0.9401	0.9167	0.9012	0.8958	
1.0024	0.9708	0.9433	0.9212	0.9067	0.9018	
1.0012	0.9703	0.9429	0.9207	0.9062	0.9012	
1.0006	0.9701	0.9427	0.9206	0.9060	0.9010	
E					F	

TABLE IIii

Problem 2: LBIE Method Results

	1.0000	0.9164	0.8310	0.7422	0.6485	0.5492
	1.0000	0.9165	0.8311	0.7422	0.6484	0.5491
	1.0000	0.9165	0.8312	0.7423	0.6485	0.5493
D						C
	1.0000	0.9173	0.8328	0.7445	0.6512	0.5519
	1.0000	0.9174	0.8329	0.7447	0.6512	0.5519
	1.0000	0.9175	0.8330	0.7448	0.6513	0.5520
	1.0000	0.9201	0.8384	0.7524	0.6601	0.5605
	1.0000	0.9203	0.8386	0.7525	0.6601	0.5602
	1.0000	0.9204	0.8387	0.7527	0.6602	0.5602
	1.0000	0.9251	0.8482	0.7666	0.6766	0.5762
	1.0000	0.9253	0.8485	0.7669	0.6769	0.5756
	1.0000	0.9254	0.8486	0.7671	0.6771	0.5755
	1.0000	0.9322	0.8627	0.7890	0.7056	0.6033
	1.0000	0.9324	0.8631	0.7896	0.7061	0.6023
	1.0000	0.9325	0.8633	0.7898	0.7065	0.6020
	1.0000	0.9408	0.8811	0.8199	0.7546	0.6658
	1.0000	0.9411	0.8817	0.8208	0.7562	0.6661
	1.0000	0.9412	0.8818	0.8210	0.7566	0.6664
						O

	0.5492	0.4456	0.3375	0.2264	0.1135	0.0000
	0.549	0.4451	0.3370	0.2260	0.1133	0.0000
	0.5493	0.4452	0.3369	0.2260	0.1133	0.0000
C						B
	0.5519	0.4478	0.3390	0.2273	0.1140	0.0000
	0.5519	0.4473	0.3384	0.2268	0.1137	0.0000
	0.5520	0.4473	0.3384	0.2268	0.1137	0.0000
	0.5605	0.4547	0.3435	0.2298	0.1151	0.0000
	0.5602	0.4537	0.3426	0.2292	0.1148	0.0000
	0.5602	0.4535	0.3424	0.2291	0.1147	0.0000
	0.5762	0.4666	0.3502	0.2332	0.1165	0.0000
	0.5756	0.4646	0.3488	0.2324	0.1161	0.0000
	0.5755	0.4642	0.3485	0.2322	0.1160	0.0000
	0.6033	0.4824	0.3571	0.2363	0.1177	0.0000
	0.6023	0.4791	0.3553	0.2353	0.1172	0.0000
	0.6020	0.4782	0.3549	0.2351	0.1172	0.0000
	0.6658	0.4897	0.3593	0.2370	0.1178	0.0000
	0.6661	0.4873	0.3580	0.2363	0.1175	0.0000
	0.6664	0.4868	0.3577	0.2362	0.1176	0.0000
O						A

1.0000	0.9500	0.9008	0.8540	0.8132	0.7959	$N = 50$
1.0000	0.9503	0.9014	0.8551	0.8152	0.7968	100
1.0000	0.9503	0.9016	0.8553	0.8156	0.7966	200
1.0000	0.9583	0.9185	0.8833	0.8573	0.8484	
1.0000	0.9585	0.9190	0.8843	0.8586	0.8490	
1.0000	0.9585	0.9191	0.8844	0.8588	0.8490	
1.0000	0.9647	0.9318	0.9041	0.8853	0.8791	
1.0000	0.9649	0.9322	0.9048	0.8861	0.8795	
1.0000	0.9649	0.9322	0.9049	0.8862	0.8795	
1.0000	0.9687	0.9398	0.9162	0.9007	0.8957	
1.0000	0.9688	0.9401	0.9167	0.9012	0.8959	
1.0000	0.9688	0.9401	0.9167	0.9013	0.8958	
1.0000	0.9699	0.9423	0.9200	0.9053	0.9007	
1.0000	0.9701	0.9427	0.9205	0.9060	0.9011	
1.0000	0.9701	0.9427	0.9206	0.9061	0.9010	

E F

TABLE IIiii

Problem 2: QBIE Method Results

	1.0000	0.9165	0.8311	0.7420	0.6480	0.5484
	1.0000	0.9165	0.8311	0.7422	0.6483	0.5490
	1.0000	0.9165	0.8312	0.7423	0.6485	0.5493
D						*C*
	1.0000	0.9174	0.8329	0.7445	0.6508	0.5512
	1.0000	0.9175	0.8330	0.7447	0.6511	0.5516
	1.0000	0.9175	0.8330	0.7448	0.6513	0.5519
	1.0000	0.9204	0.8386	0.7524	0.6598	0.5594
	1.0000	0.9204	0.8387	0.7526	0.6600	0.5599
	1.0000	0.9204	0.8387	0.7527	0.6602	0.5601
	1.0000	0.9254	0.8487	0.7670	0.6768	0.5747
	1.0000	0.9254	0.8487	0.7671	0.6770	0.5751
	1.0000	0.9254	0.8487	0.7671	0.6771	0.5754
	1.0000	0.9326	0.8635	0.7901	0.7067	0.6011
	1.0000	0.9326	0.8634	0.7900	0.7066	0.6014
	1.0000	0.9325	0.8634	0.7899	0.7066	0.6016
	1.0000	0.9414	0.8822	0.8218	0.7579	0.6659
	1.0000	0.9413	0.8820	0.8215	0.7573	0.6663
	1.0000	0.9412	0.8819	0.8212	0.7569	0.6665
						O

	0.5484	0.4442	0.3361	0.2253	0.1129	0.0000
	0.5490	0.4448	0.3366	0.2257	0.1132	0.0000
	0.5493	0.4451	0.3369	0.2259	0.1133	0.0000
C						*B*
	0.5512	0.4464	0.3375	0.2261	0.1134	0.0000
	0.5516	0.4469	0.3380	0.2265	0.1136	0.0000
	0.5519	0.4472	0.3383	0.2267	0.1137	0.0000
	0.5594	0.4526	0.3415	0.2284	0.1143	0.0000
	0.5599	0.4531	0.3420	0.2287	0.1145	0.0000
	0.5601	0.4534	0.3423	0.2289	0.1147	0.0000
	0.5747	0.4629	0.3474	0.2314	0.1156	0.0000
	0.5751	0.4635	0.3479	0.2318	0.1158	0.0000
	0.5754	0.4639	0.3483	0.2320	0.1159	0.0000
	0.6011	0.4762	0.3533	0.2340	0.1166	0.0000
	0.6014	0.4770	0.3541	0.2346	0.1169	0.0000
	0.6016	0.4776	0.3546	0.2349	0.1171	0.0000
	0.6659	0.4837	0.3557	0.2349	0.1168	0.0000
	0.6663	0.4852	0.3568	0.2357	0.1173	0.0000
	0.6665	0.4861	0.3574	0.2361	0.1174	0.0000
O						*A*

1.0000	0.9506	0.9022	0.8563	0.8173	0.7997	$N = 50$
1.0000	0.9504	0.9019	0.8559	0.8165	0.7979	100
1.0000	0.9504	0.9017	0.8555	0.8159	0.7969	200
1.0000	0.9589	0.9198	0.8854	0.8603	0.8508	
1.0000	0.9587	0.9194	0.8849	0.8595	0.8498	
1.0000	0.9586	0.9192	0.8846	0.8591	0.8492	
1.0000	0.9652	0.9328	0.9058	0.8874	0.8809	
1.0000	0.9650	0.9325	0.9053	0.8868	0.8801	
1.0000	0.9649	0.9323	0.9050	0.8864	0.8796	
1.0000	0.9691	0.9407	0.9176	0.9024	0.8971	
1.0000	0.9689	0.9404	0.9171	0.9018	0.8964	
1.0000	0.9688	0.9402	0.9168	0.9014	0.8960	
1.0000	0.9704	0.9433	0.9215	0.9072	0.9023	
1.0000	0.9702	0.9430	0.9210	0.9066	0.9016	
1.0000	0.9701	0.9428	0.9207	0.9062	0.9012	

E F

TABLE IIiv

Problem 2: MBIE Method Results

	1.0005	0.9167	0.8312	0.7423	0.6486	0.5495
	1.0001	0.9166	0.8113	0.7424	0.6487	0.5496
	1.0000	0.9166	0.8113	0.7424	0.6487	0.5496
D						C
	0.9999	0.9175	0.8331	0.7448	0.6514	0.5521
	1.0000	0.9175	0.8331	0.7449	0.6515	0.5521
	1.0000	0.9175	0.8331	0.7449	0.6515	0.5522
	1.0000	0.9204	0.8387	0.7527	0.6603	0.5603
	1.0000	0.9204	0.8387	0.7527	0.6603	0.5604
	1.0000	0.9204	0.8388	0.7528	0.6604	0.5604
	1.0000	0.9254	0.8487	0.7671	0.6772	0.5756
	1.0000	0.9254	0.8487	0.7671	0.6772	0.5756
	1.0000	0.9254	0.8487	0.7671	0.6772	0.5756
	1.0000	0.9325	0.8633	0.7898	0.7066	0.6019
	1.0000	0.9325	0.8633	0.7898	0.7066	0.6019
	1.0001	0.9325	0.8633	0.7898	0.7066	0.6019
	1.0000	0.9412	0.8818	0.8210	0.7565	0.6667
	1.0000	0.9412	0.8818	0.8211	0.7565	0.6667
	1.0001	0.9412	0.8818	0.8211	0.7565	0.6667
						O

	0.5494	0.4453	0.3371	0.2261	0.1135	0.0001
	0.5495	0.4453	0.3371	0.2261	0.1134	0.0000
	0.5496	0.4454	0.3371	0.2261	0.1134	0.0000
C						B
	0.5521	0.4473	0.3384	0.2269	0.1138	0.0000
	0.5521	0.4474	0.3385	0.2269	0.1138	0.0000
	0.5522	0.4474	0.3385	0.2269	0.1138	0.0000
	0.5603	0.4536	0.3425	0.2291	0.1147	0.0000
	0.5604	0.4536	0.3425	0.2291	0.1147	0.0000
	0.5604	0.4536	0.3425	0.2291	0.1147	0.0000
	0.5756	0.4641	0.3485	0.2322	0.1160	0.0000
	0.5756	0.4642	0.3486	0.2323	0.1161	0.0000
	0.5756	0.4642	0.3486	0.2323	0.1161	0.0000
	0.6019	0.4780	0.3549	0.2352	0.1172	0.0000
	0.6019	0.4780	0.3549	0.2352	0.1172	0.0000
	0.6019	0.4780	0.3549	0.2352	0.1172	0.0000
	0.6667	0.4869	0.3579	0.2364	0.1177	0.0000
	0.6667	0.4869	0.3579	0.2364	0.1177	0.0000
	0.6667	0.4869	0.3579	0.2364	0.1177	0.0000
O						A

1.0000	0.9503	0.9016	0.8553	0.8154	0.7962	$N = 50$
1.0001	0.9503	0.9016	0.8553	0.8154	0.7961	100
1.0001	0.9503	0.9016	0.8553	0.8154	0.7961	200
1.0001	0.9586	0.9191	0.8844	0.8587	0.8488	
1.0001	0.9586	0.9191	0.8844	0.8587	0.8487	
1.0001	0.9586	0.9191	0.8844	0.8587	0.8487	
1.0001	0.9649	0.9322	0.9049	0.8862	0.8794	
1.0001	0.9649	0.9322	0.9049	0.8862	0.8793	
1.0001	0.9649	0.9322	0.9049	0.8862	0.8793	
1.0002	0.9687	0.9401	0.9167	0.9013	0.8958	
1.0001	0.9688	0.9401	0.9167	0.9013	0.8957	
1.0001	0.9688	0.9401	0.9168	0.9014	0.8957	
0.9995	0.9699	0.9427	0.9207	0.9062	0.9010	
1.0001	0.9701	0.9428	0.9207	0.9062	0.9009	
1.0002	0.9702	0.9428	0.9207	0.9063	0.9009	

E F

The Modified BIE Method

Symm [3] showed that by including terms having the analytical form of the singularity in the CBIE method, the problems caused by the presence of the singularity can be overcome. However, this method is not directly applicable to problems in which the singularity occurs on the interface between two regions with different physical properties, because the analytical solution in the neighbourhood of the singularity is represented by different expressions in the two regions. The modifications necessary to overcome this difficulty in the case of problem 2 are now presented. The analysis for problem 1 is very similar and therefore is not presented.

First, it is necessary to determine the analytical form of the solution in the neighbourhood of the singular point, which is situated at the re-entrant corner O. Employing the polar co-ordinates (r, ξ) in region A, and (r, η) in region B, Fig. 3, the general solutions of equations (25) and (26) can be expressed as

$$\phi_A(r, \xi) = \sum_{n=0}^{\infty} r^{\lambda_n}(a_n \cos \lambda_n \xi + b_n \sin \lambda_n \xi), \tag{28}$$

$$\phi_B(r, \eta) = \sum_{n=0}^{\infty} r^{\mu_n}(c_n \cos \mu_n \eta + d_n \sin \mu_n \eta), \tag{29}$$

where the eigenvalues λ_n and μ_n, and the coefficients a_n, b_n, c_n and d_n are undetermined constants dependent upon the boundary conditions.

In the neighbourhood of the singularity, the solutions (28) and (29) are subject to the boundary conditions.

$$\text{on } \xi = 0 \qquad \phi'_A = 0, \tag{30i}$$

$$\text{on } \xi = \pi/2, \eta = \pi \qquad \phi_A = \phi_B, \tag{30ii}$$

$$\text{on } \xi = \pi/2, \eta = \pi \qquad k_A \phi'_A = -k_B \phi'_B, \tag{30iii}$$

$$\text{on } \eta = 0 \qquad \phi'_B = 0, \tag{30iv}$$

where $\xi = 0$, $\eta = 0$ and($\xi = \pi/2$, $\eta = \pi$) specify the boundaries OA, OF and OC respectively.

Enforcing conditions (30i) and (30iv), and then matching at the common interface, using conditions (30ii) and (30iii), gives

$$\phi_A(r, \xi) = \alpha + \beta r^{\lambda_1} \cos \lambda_1 \xi + \gamma r^{\lambda_2} \cos \lambda_2 \xi + \delta r^{\lambda_3} \cos \lambda_3 \xi + \cdots \tag{31}$$

and

$$\phi_B(r, \eta) = \alpha^* + \beta^* r^{\lambda_1} \cos \lambda_1 \eta + \gamma^* r^{\lambda_2} \cos \lambda_2 \eta + \delta^* r^{\lambda_3} \cos \lambda_3 \eta + \cdots, \tag{32}$$

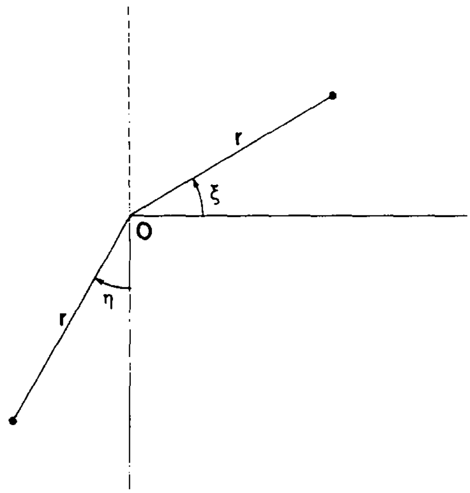

FIG. 3. Re-entrant corner neighbourhood

where α, β, γ and δ are unknown contants, and

$$\alpha^* = \alpha,$$
$$\beta^* = \beta \cos \lambda_1 \pi / \cos \lambda_1 (\pi/2),$$
$$\gamma^* = \gamma \cos \lambda_2 \pi / \cos \lambda_2 \pi,$$
$$\delta^* = -\delta,$$
$$\lambda_n = 2\varepsilon, 2(1-\varepsilon), 2, 2(1+\varepsilon), 2(2-\varepsilon), \ldots, \qquad n = 1, 2, \ldots,$$

and

$$\varepsilon = \frac{1}{\pi} \cos^{-1} \left(\frac{k_B}{2(k_A + k_B)} \right)^{1/2}.$$

Inclusion of terms of the singular solutions (31) and (32) in the CBIE method is performed by analogy with the method presented by Symm [3]. Functions ψ_A and ψ_B are defined such that

$$\phi_A(p) = \psi_A(p) + f_A(p), \qquad p \in A + \partial A, \tag{33}$$

and

$$\phi_B(p) = \psi_B(p) + f_B(p), \qquad p \in B + \partial B, \tag{34}$$

where

$$f_A(p) = \alpha + \beta r^{\lambda_1} \cos \lambda_1 \xi + \gamma r^{\lambda_2} \cos \lambda_2 \xi + \delta r^{\lambda_3} \cos \lambda_3 \xi, \qquad p = (r, \xi), \tag{35}$$

and

$$f_B(p) = \alpha^* + \beta^* r^{\lambda_1} \cos \lambda_1 \eta + \gamma^* r^{\lambda_2} \cos \lambda_2 \eta$$
$$+ \delta^* r^{\lambda_3} \cos \lambda_3 \eta, \qquad p = (r, \eta). \tag{36}$$

Thus, the functions ψ_A and ψ_B are harmonic in regions A and B, respectively, and satisfy the boundary conditions,

on OA	$\psi'_A = -f'_A$,	(37i)
on AB	$\psi_A = -f_A$,	(37ii)
on BC	$\psi'_A = -f'_A$,	(37iii)
on CO	$\psi_A = \psi_B$,	(37iv)
on CO	$k_A \psi'_A = -k_B \psi'_B$,	(37v)
on CD	$\psi'_B = -f'_B$,	(37vi)
on DE	$\psi_B = 1 - f_B$,	(37vii)
on EF	$\psi'_B = -f'_B$,	(37viii)
on FO	$\psi'_B = -f'_B$.	(37ix)

Applying the CBIE method to the functions ψ_A and ψ_B and enforcing the boundary conditions (37) generates a system of N linear algebraic euations in $N + 4$ unknowns, including the constants α, β, γ, and δ. To reduce the number of unknowns to N, it is necessary to assume that $\psi_A = 0$ on the intervals 1 and 2, Fig. 4, and $\psi_B = 0$ on the intervals N and $N - 1$, i.e., in the vicinity of the singular point O, the potentials ϕ_A and ϕ_B can be approximated by the expressions (35) and (36) for f_A and f_B. Solving

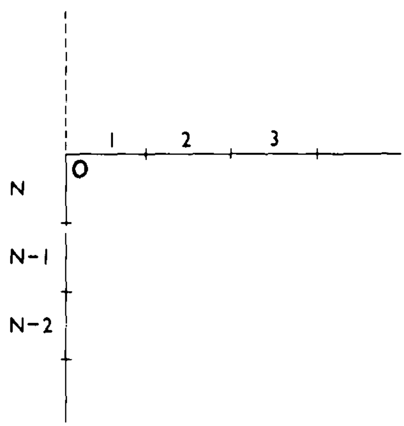

FIG. 4. Boundary discretization.

this system of equations determines the boundary distributions of ψ and ψ', and also the constants α, β, γ and δ. The potential ϕ at any point in $(A+B)$ can then be computed using appropriately discretized forms of Eq. (33) and (34).

Solutions to problem 2 have been obtained employing this modified BIE (MBIE) method, for the case $OA=AB=EF=FO=1$ and $k_A=k_B=1$, and are presented in Table IIiv. Comparison with the solutions obtained employing the standard BIE methods, Tables IIi, IIii and IIiii, shows that the MBIE method affords a considerable improvement in the rate of convergence, in particular near the singularity. However, on the boundaries AB and DE, on which the potential is prescribed to be 0 and 1, respectively, the LBIE and QBIE methods are more accurate than the MBIE method.

Problem 1 has also been solved employing the MBIE method, and representative results for the case $OA=AB=EF=FO=1$, $h_1=1000$, $h_2=10$, $k_A=250$ and $k_B=10$ are presented in Table Iiv. These results are significantly better than those given by the standard BIE methods, Table Ii–iii. In particular, the heat transfer rates, Q_{IN} and Q_{OUT}, converge appreciably more rapidly, and the requirement that the ratio of Q_{IN} and Q_{OUT} be unity is satisfied more accurately than by the CBIE, LBIE and QBIE methods.

Discussion and Conclusions

The MBIE method presented here enables effective treatment of two-dimensional Laplacian problems involving singular points at which there is also a change of the physical properties. Although the method is only described for problems involving L-shaped domains, it is applicable to any such problem for which the analytical form of the singularity can be determined. Results have been obtained for other problems and in all cases the modified BIE method facilitated an improvement in the rate of convergence.

The additional sophistication inherent in the MBIE method, while requiring considerably more programming time than the standard BIE methods, affords improved accuracy for modest boundary discretizations. Furthermore, for a given number of boundary intervals, the MBIE does not require appreciably more computational time than the CBIE and LBIE methods, and in fact only requires between one-fourth and one-half the computational time of the QBIE method. This is due to the fact that for an N interval discretization, the QBIE method generates $2N \times 2N$ equations, whereas the CBIE, LBIE and MBIE methods only generate $N \times N$ equations.

Although evaluation of the integrals associated with the LBIE and QBIE methods, by the analytical expressions presented here, requires substantially less computational time than that required by previously employed quadrature formulae [13], these analytical expressions are only applicable for rectilinear boundaries; for the two problems considered in this study it has been found that the use of the analytical expressions facilitates a reduction in the overall computational time of up to 50%, depending upon the piecewise-approximation and the size of the discretization.

However, it may be desirable to approximate curved boundaries by a series of straight-line segments, and integrate over these segments exactly.

It should be noted that the MBIE method is a modification of the CBIE method. The LBIE and QBIE methods cannot be modified, in a non-trivial way, because of the necessity to evaluate f'_A and f'_B at the point O, where these quantities are infinite. Further work on this aspect is at present under investigation.

Acknowledgments

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References

1. G. D. Smith, "Numerical Solution of Partial Differencial Equations," Oxford Univ. Press, London/New York. 1974.
2. O. C. Zienkiewicz, "The Finite Element Method in Engineering," McGraw–Hill, London, 1971.
3. G. T. Symm, "Treatment of Singularities in the Solution of Laplace's Equation by an Integral Equation Method." National Physical Laboratory, Report NAC31, 1973.
4. P. Daly, *in* "The Mathematics of Finite Elements and Applications," Brunel University, 1972.
5. J. R. Whiteman, *in* "Proceedings of the International Symposium on Finite Elements in Flow Problems," Wiley, London, 1974.
6. N. Papamichael and G. T. Symm, *Comput. Methods Appl. Mech. Eng.* **6** (1975), 175–194.
7. J. R. Whiteman and J. C. Webb, *BIT* **10** (1970), 336–374.
8. R. Wait, *J. Inst. Math. Appl.* **20** (1977), 133–141.
9. D. F. Griffiths, *J. Inst. Math. Appl.* **19** (1977), 59–69.
10. R. W. Thatcher, *J. Inst. Math. Appl.* **16** (1975), 303–319.
11. P. J. Heggs and P. R. Stones, *ASME Trans., J. Heat Transfer* **102** (1980), 180–182.
12. R. F. Harrington, K. Pontoppidan, P. Abrahemsen, and N. C. Albertsen, *Proc. IEEE* **116** (1969), 1715–1719.
13. G. Fairweather, F. J. Rizzo, D. J. Shippy and Y. S. Wu, *J. Comput. Phys.* **31** (1979), 96–112.
14. J. L. Blue, *Bell System Tech. J.* **57** (1978), 2797–2822.